Chapter 6

Variables Control Charts

Control Charts for Variables

- *Many quality characteristics can be expressed in terms of a numerical measurement.*
- *As examples, the diameter of a bearing could be measured with a micrometer and expressed in millimeters or the time to process an insurance claim can be expressed in hours.*
- *A single measurable quality characteristic, such as a dimension, weight, or volume, is called a variable.*

Control Charts for Variables

- *When dealing with a quality characteristic that is a variable, it is usually necessary to monitor both the mean value of the quality characteristic and its variability.*
- *Control of the **process average** or mean quality level is usually done with the control chart for means, or the **$\overline{x}$ control chart**.*
- ***Process variability** can be monitored with either a control chart for the standard deviation, called the **s control chart**, or a control chart for the range, called an **R control chart**.*

Control Charts for Variables

- *It is important to maintain control over both the process mean and process variability.*

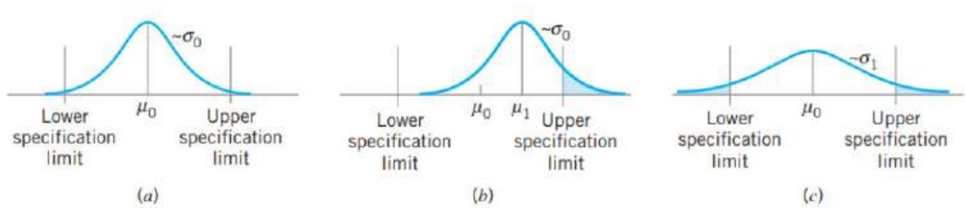


FIGURE 6.1 The need for controlling both process mean and process variability. (*a*) Mean and standard deviation at nominal levels. (*b*) Process mean $\mu_1 > \mu_0$. (*c*) Process standard deviation $\sigma_1 > \sigma_0$.

Control Charts for $\bar{x}$ and R

- *Suppose that a quality characteristic is normally distributed with mean μ and standard deviation σ, where both are known.*
- *If $x_1, x_2, \ldots, x_n$ is a sample of size n, then the average of this sample is*

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

- *and we know that $\bar{x}$ is normally distributed with mean μ and standard deviation $\sigma_{\bar{x}} = \bar{x}/\sqrt{n}$*

Control Charts for $\bar{x}$ and R

- *The probability is 1 − α that any sample mean will fall between*

$$\mu + Z_{\alpha/2}\sigma_{\bar{x}} = \mu + Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \quad \text{and} \quad \mu - Z_{\alpha/2}\sigma_{\bar{x}} = \mu - Z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \tag{6.1}$$

- *Therefore, if μ and σ are known, equation 6.1 could be used as upper and lower control limits on a control chart for sample means*

Control Charts for $\bar{x}$ and R

- *In practice, we usually will not know μ and σ.*
- *Therefore, they must be estimated from preliminary samples or subgroups taken when the process is thought to be in control.*
- *These estimates should usually be based on* **at least 20 to 25 samples**.
- *Suppose that m samples are available, each containing n observations on the quality characteristic. Typically, n will be small, often either 4, 5, or 6.*
- *Let $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ be the average of each sample.*
- *Then the best estimator of μ, the process average, is the grand average*

$$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_m}{m} \tag{6.2}$$

Control Charts for $\bar{x}$ and R

- *Thus, $\bar{\bar{x}}$ would be used as the center line on the chart*
- *To construct the control limits, we need an estimate of the standard deviation σ*
- *For the present, we will use the range method*
- *If $x_1, x_2, \ldots, x_n$ is a sample of size n, then the range of the sample is the difference between the largest and smallest observations—that is,*

$$R = x_{\max} - x_{\min}$$

- *Let $R_1, R_2, \ldots, R_m$ be the ranges of the m samples. The average range is*

$$\bar{R} = \frac{R_1 + R_2 + \cdots + R_m}{m} \tag{6.3}$$

Control Charts for $\bar{x}$ and R

- *We define the relative range as the relationship between the range of a sample from a normal distribution and the standard deviation of that distribution:*

$$W = R/\sigma$$

- *The mean of W is d_2*
- *The standard deviation of W is d_3*
- *Consequently, an estimator of σ is*

$$\hat{\sigma} = R/d_2$$

- *Values of d_2 for various sample sizes are given in Appendix Table VI*

Control Charts for $\bar{x}$ and R

- *Therefore, if $\bar{R}$ is the average range of the m preliminary samples, we may use*

$$\hat{\sigma} = \frac{\overline{R}}{d_2}$$

- *If we use $\bar{\bar{x}}$ as an estimator of μ and $\bar{\bar{R}}/d_2$ as an estimator of σ, then the parameters of the chart are*

$$A_2 = \frac{3}{d_2\sqrt{n}}$$

$$\begin{aligned} \text{UCL} &= \overline{\overline{x}} + \frac{3}{d_2\sqrt{n}}\overline{R} \\ \text{Center line} &= \overline{\overline{x}} \\ \text{LCL} &= \overline{\overline{x}} - \frac{3}{d_2\sqrt{n}}\overline{R} \end{aligned} \tag{6.7}$$

Control Charts for $\bar{x}$ and R

- *We may now give the formulas for constructing the control limits on the chart. They are as follows:*

Control Limits for the $\bar{x}$ Chart

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + A_2\bar{R} \\ \text{Center line} &= \bar{\bar{x}} \\ \text{LCL} &= \bar{\bar{x}} - A_2\bar{R} \end{aligned} \tag{6.4}$$

The constant A_2 is tabulated for various sample sizes in Appendix Table VI.

Control Charts for $\bar{x}$ and R

- *We know that*

$$R = W\sigma$$

- *And the mean of W is d_2, and the standard deviation of W is d_3*
- *So, the standard deviation of R is*

$$\sigma_R = d_3\sigma$$

- *Since σ is unknown, we may estimate σ_R by*

$$\hat{\sigma}_R = d_3\frac{\bar{R}}{d_2}$$

$$LCL = \mu_w - 3\sigma_w \xRightarrow{w=R} LCL = \mu_R - 3\sigma_R \xRightarrow{R=\hat{R}} LCL = \hat{R} - 3\sigma_{\hat{R}} \xRightarrow{\hat{R}=\bar{R}} LCL = \bar{R} - 3\sigma_{\bar{R}}$$

$$\xRightarrow{\substack{\bar{R}=d_3\hat{\sigma} \\ R=W\sigma}} UCL = \bar{R} - 3d_3\hat{\sigma} \xRightarrow{\hat{\sigma}=\frac{\bar{R}}{d_2}} LCL = \bar{R} - 3d_3\frac{\bar{R}}{d_2} = \bar{R}\left(1 - 3d_3\frac{1}{d_2}\right)$$

$$\xRightarrow{D_3=\left(1-\frac{3d_3}{d_2}\right)} LCL = \bar{R}D_3$$

Control Charts for $\bar{x}$ and R

- *Consequently, the parameters of the R chart with the usual three-sigma control limits are*

$$\begin{aligned} \text{UCL} &= \overline{R} + 3\hat{\sigma}_R = \overline{R} + 3d_3\frac{\overline{R}}{d_2} \\ \text{Center line} &= \overline{R} \\ \text{LCL} &= \overline{R} - 3\hat{\sigma}_R = \overline{R} - 3d_3\frac{\overline{R}}{d_2} \end{aligned} \tag{6.10}$$

$$D_3 = 1 - 3\frac{d_3}{d_2} \quad \text{and} \quad D_4 = 1 + 3\frac{d_3}{d_2}$$

Control Charts for $\bar{x}$ and R

- *Process variability may be monitored by plotting values of the sample range R on a control chart. The center line and control limits of the R chart are as follows:*

Control Limits for the *R* Chart

$$\mathrm{UCL} = D_4\overline{R}$$
$$\text{Center line} = \overline{R} \qquad (6.5)$$
$$\mathrm{LCL} = D_3\overline{R}$$

The constants D_3 and D_4 are tabulated for various values of n in Appendix Table VI.

APPENDIX VI

Factors for Constructing Variables Control Charts

Observations in Sample, n	Chart for Averages: Factors for Control Limits			Chart for Averages: Factors for Center Line		Chart for Standard Deviations: Factors for Control Limits				Chart for Standard Deviations: Factors for Center Line		Chart for Ranges: Factors for Control Limits				
	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.574
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541

Development and Use of $\bar{x}$ and R Charts

- *Example 6.1. A hard-bake process is used in conjunction with photolithography in semiconductor manufacturing.*
  We wish to establish statistical control of the flow width of the resist in this process using $\bar{x}$ and R charts.
  Twenty-five samples, each of size five wafers, have been taken when we think the process is in control.
  The interval of time between samples or subgroups is one hour. The flow width measurement data (in microns) from these samples are shown in Table 6.1.

Development and Use of $\bar{x}$ and R Charts

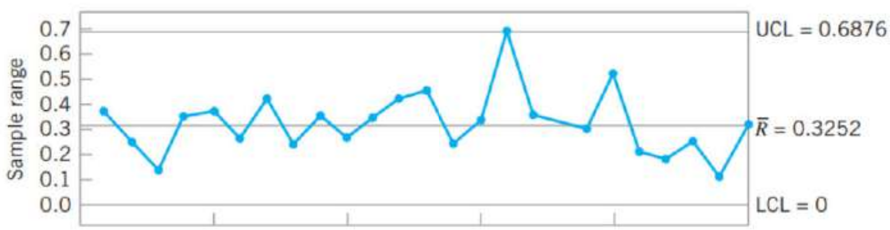
- *Solution 6.1.*
- *When setting up and R control charts, it is best to **begin with the R chart**.*
- *Because the control limits on the $\bar{x}$ chart depend on the process variability, unless process variability is in control, these limits will not have much meaning*
- *Using the data in Table 6.1, we find that the center line for the R chart*

$$\overline{R} = \frac{\sum_{i=1}^{25} R_i}{25} = \frac{8.1302}{25} = 0.32521$$

Development and Use of $\bar{x}$ and R Charts

- *For samples of $n = 5$, we find from Appendix Table VI that $D_3 = 0$ and $D_4 = 2.114$.*

$$\text{LCL} = \overline{R}D_3 = 0.32521(0) = 0$$
$$\text{UCL} = \overline{R}D_4 = 0.32521(2.114) = 0.68749$$



Development and Use of $\bar{x}$ and R Charts

- *Since the R chart indicates that process variability is in control, we may now construct the $\bar{x}$ chart. The center line is*

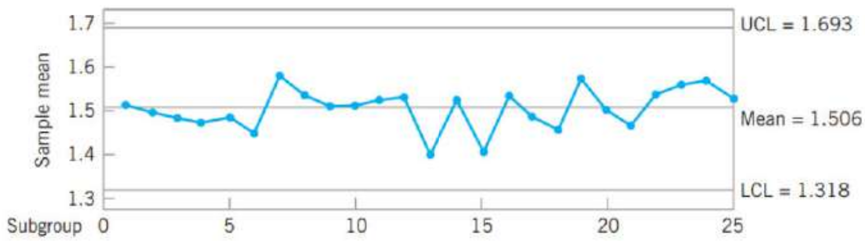
$$\bar{\bar{x}} = \frac{\sum_{i=1}^{25} \bar{x}_i}{25} = \frac{37.6400}{25} = 1.5056$$

- *To find the control limits on the chart, we use $A_2 = 0.577$ from Appendix Table VI for samples of size $n = 5$ and equation 6.4 to find*

$$\text{UCL} = \bar{\bar{x}} + A_2\overline{R} = 1.5056 + (0.577)(0.32521) = 1.69325$$
$$\text{LCL} = \bar{\bar{x}} - A_2\overline{R} = 1.5056 - (0.577)(0.32521) = 1.31795$$

Development and Use of $\bar{x}$ and R Charts

- *When the preliminary sample averages are plotted on this chart, no indication of an out-of-control condition is observed*



(a)

Other $\bar{x}$ and R Charts

- *We have two other special cases:*
  - *Changing Sample Size on the $\bar{x}$ and R Charts*
  - *Charts Based on Standard Values*

Changing Sample Size on the $\bar{x}$ and R Charts

- *We have presented the development of $\bar{x}$ and R charts assuming that the sample size n is constant from sample to sample.*
- *One situation is that of* ***variable sample size on control charts****; that is, each sample may consist of a different number of observations.*
- *Another situation is that of making a* **permanent** *(or semipermanent) change in the sample size.*

Changing Sample Size on the $\bar{x}$ and R Charts

- *For the $\bar{x}$ chart the new control limits are*

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\bar{R}_{\text{old}} \\ \text{LCL} &= \bar{\bar{x}} - A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\bar{R}_{\text{old}} \end{aligned} \tag{6.12}$$

Changing Sample Size on the $\bar{x}$ and R Charts

- *For the R chart, the new parameters are*

$$
\begin{aligned}
\text{UCL} &= D_4\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\bar{R}_{\text{old}} \\
\text{CL} = \bar{R}_{\text{new}} &= \left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\bar{R}_{\text{old}} \\
\text{LCL} &= \max\left\{0, D_3\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\bar{R}_{\text{old}}\right\}
\end{aligned}
\qquad (6.13)
$$

Changing Sample Size on the $\bar{x}$ and R Charts

- *Example 6.2. To illustrate the above procedure, consider the $\bar{x}$ and R charts developed for the hard-bake process in Example 6.1.*
- *These charts were based on a sample size of **five** wafers.*
- *Suppose that since the process exhibits good control, the process engineering personnel want to reduce the sample size to **three** wafers.*
- *Set up the new control charts.*

Changing Sample Size on the $\bar{x}$ and R Charts

- *Solution 6.2.*
- *From Example 6.1, we know that*
- *and from Appendix Table VI we have* $n_{\text{old}} = 5 \quad \overline{R}_{\text{old}} = 0.32521$
- *Therefore, the new control limits on the $\bar{x}$ chart are found from equation 6.12 as*

$$d_2(\text{old}) = 2.326 \qquad d_2(\text{new}) = 1.693$$

$$\begin{aligned}\text{UCL} &= \bar{\bar{x}} + A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}\\ &= 1.5056 + (1.023)\left[\frac{1.693}{2.326}\right](0.32521)\\ &= 1.5056 + 0.2422 = 1.7478\end{aligned}$$

$$\begin{aligned}\text{LCL} &= \bar{\bar{x}} - A_2\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}\\ &= 1.5056 - (1.023)\left[\frac{1.693}{2.326}\right](0.32521)\\ &= 1.5056 - 0.2422 = 1.2634\end{aligned}$$

Changing Sample Size on the $\bar{x}$ and R Charts

- *Solution 6.2.*
- *For the R chart, the new parameters are given by equation 6.13:*

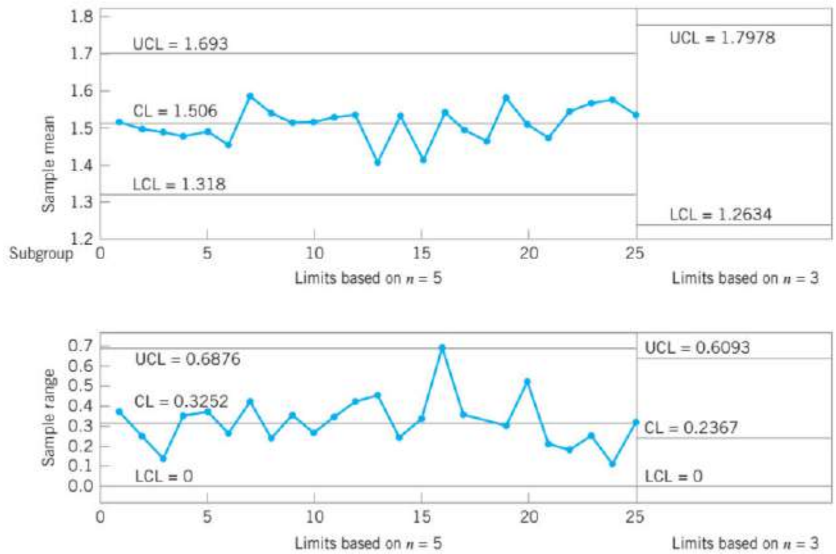
$$\begin{aligned}\text{UCL} &= D_4\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}\\ &= (2.574)\left[\frac{1.693}{2.326}\right](0.32521)\\ &= 0.6093\end{aligned}$$

$$\begin{aligned}\text{CL} = \overline{R}_{\text{new}} &= \left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}\\ &= \left[\frac{1.693}{2.326}\right](0.32521)\\ &= 0.2367\end{aligned}$$

$$\begin{aligned}\text{LCL} &= \max\left\{0, D_3\left[\frac{d_2(\text{new})}{d_2(\text{old})}\right]\overline{R}_{\text{old}}\right\}\\ &= 0\end{aligned}$$

Changing Sample Size on the $\bar{x}$ and R Charts

- Solution 6.2.



Charts Based on Standard Values

- When it is possible to specify standard values for the process mean and standard deviation
- Suppose that the standards given are μ and σ.
- Then the parameters of the $\bar{x}$ chart are

$$
\begin{aligned}
\text{UCL} &= \mu + A\sigma \\
\text{Center line} &= \mu \\
\text{LCL} &= \mu - A\sigma
\end{aligned}
\tag{6.15}
$$

- The quantity $3/\sqrt{n}$, is a constant that depends on n, which has been tabulated in Appendix Table VI

Charts Based on Standard Values

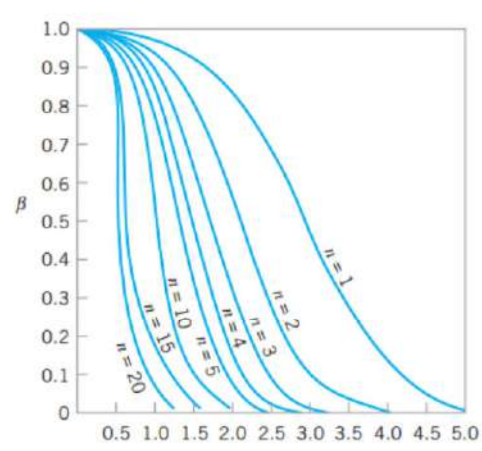
- *To construct the R chart with a standard value of σ,*
- *Then the parameters of the R chart are*

$$\begin{aligned} \text{UCL} &= D_2\sigma \\ \text{Center line} &= d_2\sigma \\ \text{LCL} &= D_1\sigma \end{aligned} \tag{6.17}$$

- *These constants are tabulated in Appendix Table VI*

The Operating-Characteristic Function

- *The ability of the $\bar{x}$ and R charts to detect shifts in process quality is described by their Operating-Characteristic (OC) curves.*
- *These OC curves for charts used for phase II monitoring of a process.*



OC Curve for $\bar{x}$ Control Chart

- *Suppose the standard deviation σ is assumed known and constant.*
- *If the mean shifts from the in-control value to another value equals with*

$$\mu_1 = \mu_0 + k\sigma$$

*the probability of **not** detecting this shift on the first subsequent sample or the β-risk is*

$$\beta = P\{\text{LCL} \le \bar{x} \le \text{UCL} \mid \mu = \mu_1 = \mu_0 + k\sigma\} \qquad (6.18)$$

OC Curve for $\bar{x}$ Control Chart

Since

and the $\bar{x} \sim N(\mu, \sigma^2/n)$ *control limits are*

$$\text{UCL} = \mu_0 + L\sigma/\sqrt{n}$$

$$\text{LCL} = \mu_0 - L\sigma/\sqrt{n}.$$

we may write equation 6.18 as

$$\beta = \Phi\left[\frac{\text{UCL} - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}}\right] - \Phi\left[\frac{\text{LCL} - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}}\right]$$

$$= \Phi\left[\frac{\mu_0 + L\sigma/\sqrt{n} - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}}\right] - \Phi\left[\frac{\mu_0 - L\sigma/\sqrt{n} - (\mu_0 + k\sigma)}{\sigma/\sqrt{n}}\right]$$

OC Curve for $\bar{x}$ Control Chart

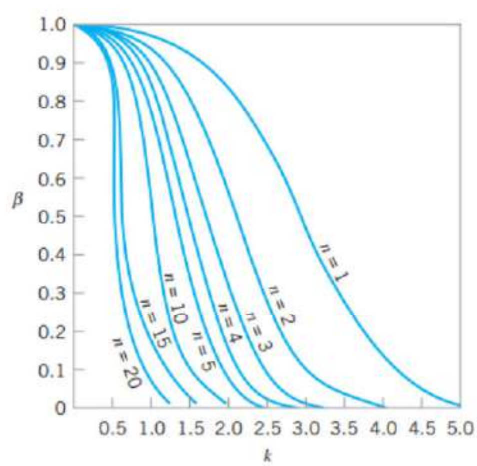
The equation reduces to

$$\beta = \Phi\left(L - k\sqrt{n}\right) - \Phi\left(-L - k\sqrt{n}\right) \qquad (6.19)$$

- *This is the β-risk, or the probability of not detecting such a shift.*
- *The probability that such a shift **will be detected** on the first subsequent sample is $1 - \beta$*

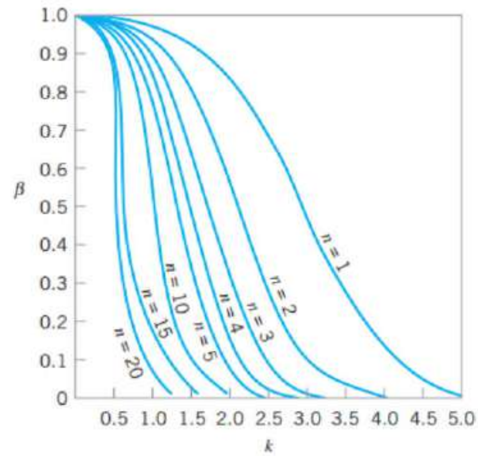
OC Curve for $\bar{x}$ Control Chart

- *To construct the OC curve for the chart, we plot the β-risk against the magnitude of the shift we wish to detect expressed in standard deviation units for various sample sizes n.*
- *These probabilities may be evaluated directly from equation 6.19.*
- *This OC curve is shown in Figure 6.13 for the case of three-sigma limits (L = 3).*



OC Curve for $\bar{x}$ Control Chart

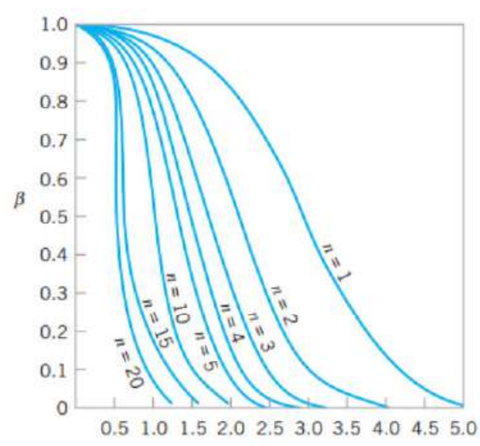
- *Consider there is a 1.0σ shift in process*
- *And your sample size is 5, n=5*
- *From figure, β=0.75, approximately.*
- *Thus, the probability that the shift will be detected on the first sample is only* $1 - \beta = 0.25$
- *The probability that the shift is detected on the second sample is* $\beta(1 - \beta) = 0.75(0.25) = 0.19$



OC Curve for $\bar{x}$ Control Chart

- *Thus, the probability that the shift will be detected on the* r^{th} *subsequent sample is simply* $1 - \beta$ *times the probability of not detecting the shift on each of the initial* $r - 1$ *samples, or*

$$\beta^{r-1}(1-\beta)$$



OC Curve for $\bar{x}$ Control Chart

- *In general, the expected number of samples taken before the shift is detected is simply the average run length, or*

$$\text{ARL} = \sum_{r=1}^{\infty} r\beta^{r-1}(1-\beta) = \frac{1}{1-\beta}$$

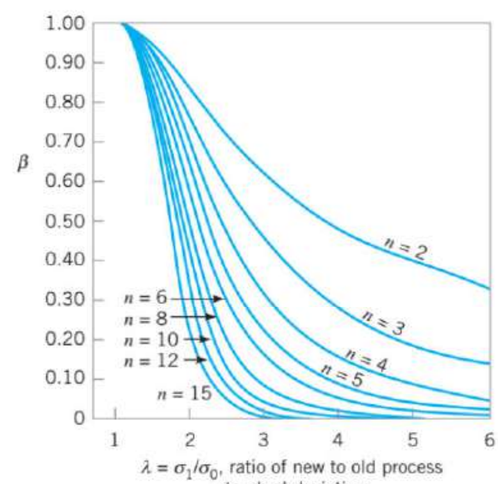
- *Therefore, in our example, we have*

$$\text{ARL} = \frac{1}{1-\beta} = \frac{1}{0.25} = 4$$

- *In other words, the expected number of samples taken to detect a shift of 1.0σ with n = 5 is **four***

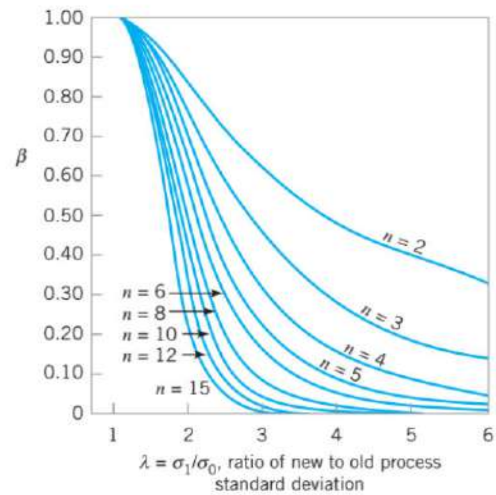
OC Curve for R Control Chart

- *Suppose that the in-control value of the standard deviation is σ_0*
- *Then the OC curve plots the probability of **not** detecting a shift to a new value of σ, $\sigma_1 > \sigma_0$ on the first sample following the shift.*
- *Figure presents the OC curve, in which β is plotted against $\lambda = \sigma_1/\sigma_0$ (the ratio of new to old process standard deviation) for various values of n.*



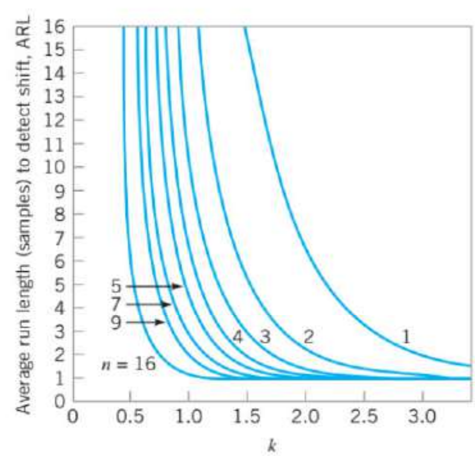
OC Curve for R Control Chart

- *Consider $\lambda=\sigma_1/\sigma_0=2$*
- *Samples of size 5, n=5*
- *The chance of detecting this shift on the subsequent sample is about 40%*



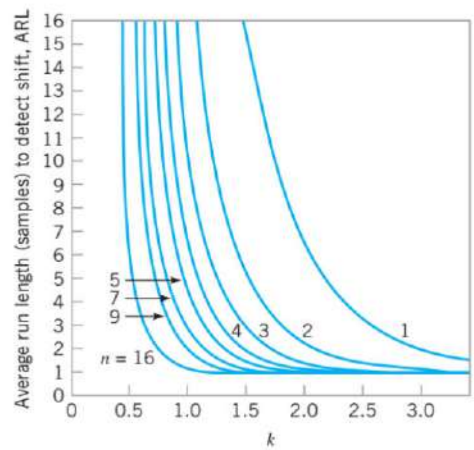
The Average Run Length for $\bar{x}$ Chart

- *The average run length to detect a shift in the mean of kσ*
- *Figure presents the ARL curves for sample sizes of n=1, 2, 3, 4, 5, 7, 9, 16 for the control chart, where the ARL is in terms of the* ***expected number of samples taken in order to detect the shift***



The Average Run Length for $\bar{x}$ Chart

- Consider that we want to detect a shift of 1.5σ using a sample size of $n=3$
- So, the average number of samples required will be 3
- Now, consider a case that we have $n=16$
- We will detect the shift of 1.5σ in 1 sample



The Average Run Length for $\bar{x}$ Chart

- Two other performance measures based on ARL are sometimes of interest.
    - ATS, average time to signal
    - I, expected number of individual units
- The average time to signal is the number of time periods that occur until a signal is generated on the control chart.
- If samples are taken at equally spaced intervals of time h, then the average time to signal or the ATS is

$$\text{ATS} = \text{ARL}\, h \tag{6.22}$$

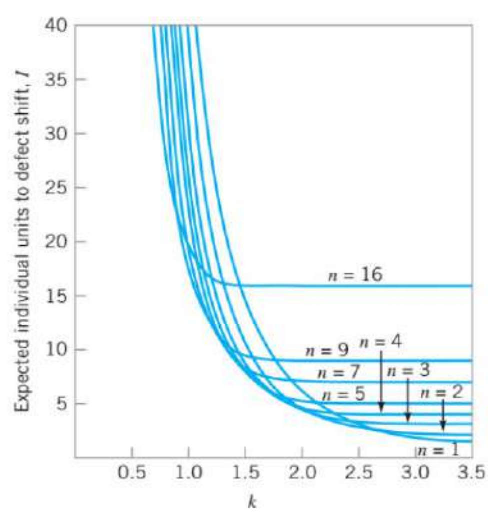
The Average Run Length for $\bar{x}$ Chart

- *Two other performance measures based on ARL are sometimes of interest.*
  - *ATS, average time to signal*
  - *I, expected number of individual units*
- *It may also be useful to express the ARL in terms of the expected number of individual units sampled, rather than the number of samples taken to detect a shift.*
- *If the sample size is n, the relationship between I and ARL is*

$$I = n\ \text{ARL} \qquad (6.23)$$

The Individual Unit for $\bar{x}$ Chart

- *Figure presents a set of curves that plot the expected number of individual units I that must be sampled for the chart to detect a shift of kσ.*
- *To illustrate the use of the curve, note that to detect a shift of 1.5σ, an $\bar{x}$ chart with n=16 will require that 16 units be sampled*
- *Whereas if the sample size is n=3, only about 9 units will be required, on the average.*



Control Charts for $\bar{x}$ and s

- *Generally, $\bar{x}$ and s charts are preferable to their more familiar counterparts, $\bar{x}$ and R charts, when either*
    1. *The sample size n is moderately large—say, n > 10 or 12*
    2. *The sample size n is variable*
- *If σ^2 is the unknown variance of a probability distribution, then an unbiased estimator of σ^2 is the sample variance*

$$s^2 = \frac{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}{n-1}$$

Control Charts for $\bar{x}$ and s

- *Consider the case where a standard value is given for σ*
- *We know that $\mu_s = E(s) = c_4\sigma$ and $\sigma_s = \sqrt{Var(s)} = \sigma\sqrt{1 - c_4^2}$*
- *The three-sigma control limits for s are then*

$$\mathrm{UCL} = c_4\sigma + 3\sigma\sqrt{1 - c_4^2}$$

$$\mathrm{LCL} = c_4\sigma - 3\sigma\sqrt{1 - c_4^2}$$

- *It is customary to define the two constants*

$$B_5 = c_4 - 3\sqrt{1 - c_4^2} \quad \text{and} \quad B_6 = c_4 + 3\sqrt{1 - c_4^2}$$

Control Charts for $\bar{x}$ and s

- *Consequently, the parameters of the s chart with a standard value for σ given become*

$$\begin{aligned} \text{UCL} &= B_6\sigma \\ \text{Center line} &= c_4\sigma \\ \text{LCL} &= B_5\sigma \end{aligned} \qquad (6.25)$$

- *Values of B_5 and B_6 are tabulated for various sample sizes in Appendix Table VI*

Control Charts for $\bar{x}$ and s

- *If no standard is given for σ, then it must be estimated by analyzing past data*
- *Suppose that m preliminary samples are available, each of size n, and let s_i be the standard deviation of the i^{th} sample.*
- *The average of the m standard deviations is*

$$\bar{s} = \frac{1}{m}\sum_{i=1}^{m} s_i$$

Control Charts for $\bar{x}$ and s

- *The statistic $\bar{s}/c_4$ is an unbiased estimator of σ, or $E(\hat{\sigma}) = \bar{s}/c_4$.*
- *Therefore, the parameters of the s chart would be*

$$\text{UCL} = \bar{s} + 3\frac{\bar{s}}{c_4}\sqrt{1-c_4^2}$$
$$\text{Center line} = \bar{s}$$
$$\text{LCL} = \bar{s} - 3\frac{\bar{s}}{c_4}\sqrt{1-c_4^2}$$

- *We usually define the constants*

$$B_3 = 1 - \frac{3}{c_4}\sqrt{1-c_4^2} \quad \text{and} \quad B_4 = 1 + \frac{3}{c_4}\sqrt{1-c_4^2}$$

Control Charts for $\bar{x}$ and s

- *Consequently, we may write the parameters of the s chart as*

$$\begin{aligned} \text{UCL} &= B_4\bar{s} \\ \text{Center line} &= \bar{s} \\ \text{LCL} &= B_3\bar{s} \end{aligned} \qquad (6.27)$$

- *Values of B_3 and B_4 are tabulated for various sample sizes in Appendix Table VI*

Control Charts for $\bar{x}$ and s

- *Since we used $\bar{s}/c_4$ is an unbiased estimator of σ, we may define the control limits on the corresponding $\bar{x}$ chart as*

$$\text{UCL} = \bar{\bar{x}} + \frac{3\bar{s}}{c_4\sqrt{n}}$$
$$\text{Center line} = \bar{\bar{x}}$$
$$\text{LCL} = \bar{\bar{x}} - \frac{3\bar{s}}{c_4\sqrt{n}}$$

- *Let the constant* $A_3 = 3/(c_4\sqrt{n})$

Control Charts for $\bar{x}$ and s

- *Then the $\bar{x}$ chart parameters become*

$$\begin{aligned} \text{UCL} &= \bar{\bar{x}} + A_3\bar{s} \\ \text{Center line} &= \bar{\bar{x}} \\ \text{LCL} &= \bar{\bar{x}} - A_3\bar{s} \end{aligned} \tag{6.28}$$

- *Value of A_3 is tabulated for various sample sizes in Appendix Table VI*

Control Charts for $\bar{x}$ and s

- *Example 6.3. Construct and interpret and s charts using the piston ring inside diameter measurements in Table 6.3.*

TABLE 6.3
Inside Diameter Measurements (mm) for Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	s_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001	74.011	74.004	74.001	0.0075
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985	73.993	73.996	0.0087
7	73.995	74.006	73.994	74.000	74.005	74.000	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005	74.004	74.004	0.0055
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998	73.997	74.012	73.998	0.0105
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998	73.999	74.007	74.006	0.0073
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005	74.007	74.001	0.0106
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013	74.020	74.003	74.009	0.0080
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162
						$\Sigma = 1{,}850.028$	0.2351
						$\bar{\bar{x}} = 74.001$	$\bar{s} = 0.0094$

Control Charts for $\bar{x}$ and s

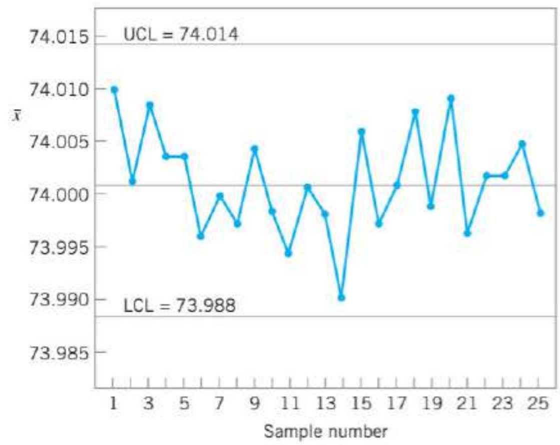
- *Solution 6.3.*
- *The grand average and the average standard deviation are*

$$\bar{\bar{x}} = \frac{1}{25}\sum_{i=1}^{25}\bar{x}_i = \frac{1}{25}(1{,}850.028) = 74.001 \qquad \bar{s} = \frac{1}{25}\sum_{i=1}^{25}s_i = \frac{1}{25}(0.2351) = 0.0094$$

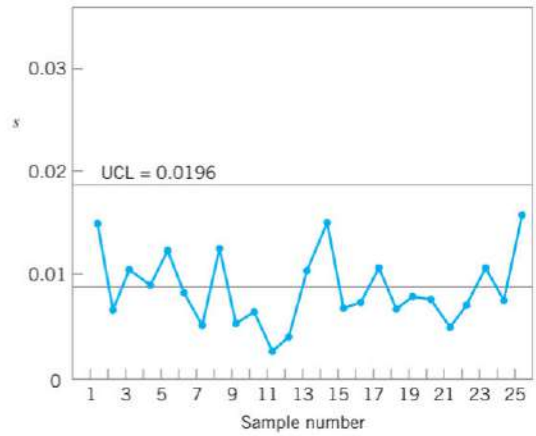
$$\begin{aligned}
\text{UCL} &= \bar{\bar{x}} + A_3\bar{s} = 74.001 + (1.427)(0.0094) = 74.014 \\
\text{CL} &= \bar{\bar{x}} = 74.001 \\
\text{LCL} &= \bar{\bar{x}} - A_3\bar{s} = 74.001 - (1.427)(0.0094) = 73.988
\end{aligned}$$

$$\begin{aligned}
\text{UCL} &= B_4\bar{s} = (2.089)(0.0094) = 0.0196 \\
\text{CL} &= \bar{s} = 0.0094 \\
\text{LCL} &= B_3\bar{s} = (0)(0.0094) = 0
\end{aligned}$$

Control Charts for $\bar{x}$ and s



(a)



(b)

$\bar{x}$ and s Control Charts with Variable Sample Size

- *In this case, we should use a weighted average approach in calculating $\bar{\bar{x}}$ and $\bar{s}$*
- *If n_i is the number of observations in the i^{th} sample, then use*

$$\bar{\bar{x}} = \frac{\sum_{i=1}^{m} n_i \bar{x}_i}{\sum_{i=1}^{m} n_i} \qquad \bar{s} = \left[\frac{\sum_{i=1}^{m} (n_i - 1) s_i^2}{\sum_{i=1}^{m} n_i - m} \right]^{1/2}$$

as the center lines on the $\bar{x}$ and s control charts, respectively.

- *The control limits would be calculated from equations 6.27 and 6.28, respectively, but the constants A_3, B_3, and B_4 will depend on the sample size used* ***in each individual subgroup****.*

$\bar{x}$ and s Control Charts w

- *Example 6.4. Consider the data in Table 6.4, which is a modification of the piston-ring data used in Example 6.3.*
- *Note that the sample sizes vary from n = 3 to n = 5.*
- *Set up the $\bar{x}$ and s control charts.*

TABLE 6.4

Inside Diameter Measurements (mm) on Automobile Engine Piston Rings

Sample Number	Observations					$\bar{x}_i$	s_i
1	74.030	74.002	74.019	73.992	74.008	74.010	0.0148
2	73.995	73.992	74.001			73.996	0.0046
3	73.988	74.024	74.021	74.005	74.002	74.008	0.0147
4	74.002	73.996	73.993	74.015	74.009	74.003	0.0091
5	73.992	74.007	74.015	73.989	74.014	74.003	0.0122
6	74.009	73.994	73.997	73.985		73.996	0.0099
7	73.995	74.006	73.994	74.000		73.999	0.0055
8	73.985	74.003	73.993	74.015	73.988	73.997	0.0123
9	74.008	73.995	74.009	74.005		74.004	0.0064
10	73.998	74.000	73.990	74.007	73.995	73.998	0.0063
11	73.994	73.998	73.994	73.995	73.990	73.994	0.0029
12	74.004	74.000	74.007	74.000	73.996	74.001	0.0042
13	73.983	74.002	73.998			73.994	0.0100
14	74.006	73.967	73.994	74.000	73.984	73.990	0.0153
15	74.012	74.014	73.998			74.008	0.0087
16	74.000	73.984	74.005	73.998	73.996	73.997	0.0078
17	73.994	74.012	73.986	74.005		73.999	0.0115
18	74.006	74.010	74.018	74.003	74.000	74.007	0.0070
19	73.984	74.002	74.003	74.005	73.997	73.998	0.0085
20	74.000	74.010	74.013			74.008	0.0068
21	73.982	74.001	74.015	74.005	73.996	74.000	0.0122
22	74.004	73.999	73.990	74.006	74.009	74.002	0.0074
23	74.010	73.989	73.990	74.009	74.014	74.002	0.0119
24	74.015	74.008	73.993	74.000	74.010	74.005	0.0087
25	73.982	73.984	73.995	74.017	74.013	73.998	0.0162

$\bar{x}$ and s Control Charts with Variable Sample Size

- *Solution 6.4.*
- *The weighted grand mean and weighted average standard deviation are computed from equations 6.30 and 6.31 as follows:*

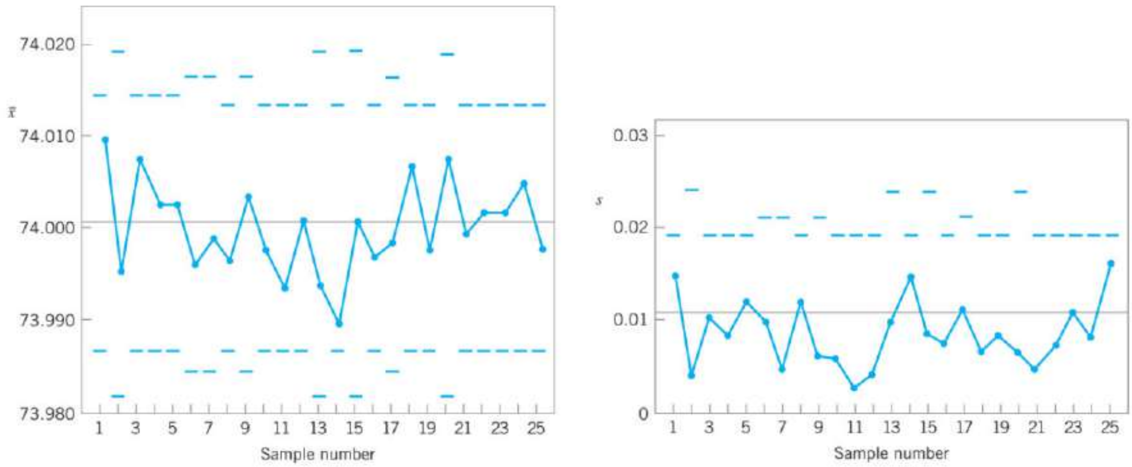
$$\bar{\bar{x}} = \frac{\sum_{i=1}^{25} n_i \bar{x}_i}{\sum_{i=1}^{25} n_i} = \frac{5(74.010) + 3(73.996) + \cdots + 5(73.998)}{5 + 3 + \cdots + 5}$$

$$= \frac{8{,}362.075}{113} = 74.001$$

$$\bar{s} = \left[\frac{\sum_{i=1}^{25} (n_i - 1) s_i^2}{\sum_{i=1}^{25} n_i - 25} \right]^{1/2} = \left[\frac{4(0.0148)^2 + 2(0.0046)^2 + \cdots + 4(0.0162)^2}{5 + 3 + \cdots + 5 - 25} \right]^{1/2}$$

$$= \left[\frac{0.009324}{88} \right]^{1/2} = 0.0103$$

$\bar{x}$ and s Control Charts with Variable Sample Size



(a) (b)

s^2 Control Charts

- *Most quality engineers use either the R chart or the s chart to monitor process variability, with s preferable to R for moderate to large sample sizes.*
- *Some practitioners recommend a control chart based directly on the sample variance s^2, the **s^2 control chart**.*
- *The parameters for the s^2 control chart are*

$$\begin{aligned} \text{UCL} &= \frac{\bar{s}^2}{n-1}\chi^2_{\alpha/2,n-1} \\ \text{Center line} &= \bar{s}^2 \\ \text{LCL} &= \frac{\bar{s}^2}{n-1}\chi^2_{1-(\alpha/2),n-1} \end{aligned} \qquad (6.32)$$

Control Charts

Formulas for Control Charts, Standards Given

Chart	Center Line	Control Limits
$\bar{x}$ (μ and σ given)	μ	$\mu \pm A\sigma$
R (σ given)	$d_2\sigma$	$\text{UCL} = D_2\sigma$, $\text{LCL} = D_1\sigma$
s (σ given)	$c_4\sigma$	$\text{UCL} = B_6\sigma$, $\text{LCL} = B_5\sigma$

Formulas for Control Charts, Control Limits Based on Past Data (No Standards Given)

Chart	Center Line	Control Limits
$\bar{x}$ (using R)	$\bar{\bar{x}}$	$\bar{\bar{x}} \pm A_2\bar{R}$
$\bar{x}$ (using s)	$\bar{\bar{x}}$	$\bar{\bar{x}} \pm A_3S$
R	$\bar{R}$	$\text{UCL} = D_4\bar{R}$, $\text{LCL} = D_3\bar{R}$
s	$\bar{s}$	$\text{UCL} = B_4\bar{s}$, $\text{LCL} = B_3\bar{s}$

APPENDIX VI

Factors for Constructing Variables Control Charts

Observations in Sample, n	Chart for Averages: Factors for Control Limits			Chart for Averages: Factors for Center Line		Chart for Standard Deviations: Factors for Control Limits				Chart for Ranges: Factors for Center Line		Chart for Ranges: Factors for Control Limits				
	A	A_2	A_3	c_4	$1/c_4$	B_3	B_4	B_5	B_6	d_2	$1/d_2$	d_3	D_1	D_2	D_3	D_4
2	2.121	1.880	2.659	0.7979	1.2533	0	3.267	0	2.606	1.128	0.8865	0.853	0	3.686	0	3.267
3	1.732	1.023	1.954	0.8862	1.1284	0	2.568	0	2.276	1.693	0.5907	0.888	0	4.358	0	2.574
4	1.500	0.729	1.628	0.9213	1.0854	0	2.266	0	2.088	2.059	0.4857	0.880	0	4.698	0	2.282
5	1.342	0.577	1.427	0.9400	1.0638	0	2.089	0	1.964	2.326	0.4299	0.864	0	4.918	0	2.114
6	1.225	0.483	1.287	0.9515	1.0510	0.030	1.970	0.029	1.874	2.534	0.3946	0.848	0	5.078	0	2.004
7	1.134	0.419	1.182	0.9594	1.0423	0.118	1.882	0.113	1.806	2.704	0.3698	0.833	0.204	5.204	0.076	1.924
8	1.061	0.373	1.099	0.9650	1.0363	0.185	1.815	0.179	1.751	2.847	0.3512	0.820	0.388	5.306	0.136	1.864
9	1.000	0.337	1.032	0.9693	1.0317	0.239	1.761	0.232	1.707	2.970	0.3367	0.808	0.547	5.393	0.184	1.816
10	0.949	0.308	0.975	0.9727	1.0281	0.284	1.716	0.276	1.669	3.078	0.3249	0.797	0.687	5.469	0.223	1.777
11	0.905	0.285	0.927	0.9754	1.0252	0.321	1.679	0.313	1.637	3.173	0.3152	0.787	0.811	5.535	0.256	1.744
12	0.866	0.266	0.886	0.9776	1.0229	0.354	1.646	0.346	1.610	3.258	0.3069	0.778	0.922	5.594	0.283	1.717
13	0.832	0.249	0.850	0.9794	1.0210	0.382	1.618	0.374	1.585	3.336	0.2998	0.770	1.025	5.647	0.307	1.693
14	0.802	0.235	0.817	0.9810	1.0194	0.406	1.594	0.399	1.563	3.407	0.2935	0.763	1.118	5.696	0.328	1.672
15	0.775	0.223	0.789	0.9823	1.0180	0.428	1.572	0.421	1.544	3.472	0.2880	0.756	1.203	5.741	0.347	1.653
16	0.750	0.212	0.763	0.9835	1.0168	0.448	1.552	0.440	1.526	3.532	0.2831	0.750	1.282	5.782	0.363	1.637
17	0.728	0.203	0.739	0.9845	1.0157	0.466	1.534	0.458	1.511	3.588	0.2787	0.744	1.356	5.820	0.378	1.622
18	0.707	0.194	0.718	0.9854	1.0148	0.482	1.518	0.475	1.496	3.640	0.2747	0.739	1.424	5.856	0.391	1.608
19	0.688	0.187	0.698	0.9862	1.0140	0.497	1.503	0.490	1.483	3.689	0.2711	0.734	1.487	5.891	0.403	1.597
20	0.671	0.180	0.680	0.9869	1.0133	0.510	1.490	0.504	1.470	3.735	0.2677	0.729	1.549	5.921	0.415	1.585
21	0.655	0.173	0.663	0.9876	1.0126	0.523	1.477	0.516	1.459	3.778	0.2647	0.724	1.605	5.951	0.425	1.575
22	0.640	0.167	0.647	0.9882	1.0119	0.534	1.466	0.528	1.448	3.819	0.2618	0.720	1.659	5.979	0.434	1.566
23	0.626	0.162	0.633	0.9887	1.0114	0.545	1.455	0.539	1.438	3.858	0.2592	0.716	1.710	6.006	0.443	1.557
24	0.612	0.157	0.619	0.9892	1.0109	0.555	1.445	0.549	1.429	3.895	0.2567	0.712	1.759	6.031	0.451	1.548
25	0.600	0.153	0.606	0.9896	1.0105	0.565	1.435	0.559	1.420	3.931	0.2544	0.708	1.806	6.056	0.459	1.541